Second Orthogonality Relations

conjugacy classes in G

character table of G

irreducible characters

	g_1	g_2	$\cdots$	g_s
χ_1	$\chi_1(g_1)$	$\chi_1(g_2)$	$\cdots$	$\chi_1(g_s)$
χ_2	$\chi_2(g_1)$	$\chi_2(g_2)$	$\cdots$	$\chi_2(g_s)$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
χ_s	$\chi_s(g_1)$	$\chi_s(g_2)$	$\cdots$	$\chi_s(g_s)$

$U \in Mat_{s\times s}(\mathbb{C})$ is **unitary** if

- rows of U are an O.N.B of $\mathbb{C}^n$

$\Updownarrow$

- $UU^* = I$ $\qquad (U^* = \overline{U}^T)$

$\Updownarrow$

- $U^*U = I$

$\Updownarrow$

- columns of U are an O.N.B. of $\mathbb{C}^n$

Lemma: Define $\underset{\substack{\parallel \\ (u_{ij})}}{U} \in \mathrm{Mat}_{s\times s}(\mathbb{C})$

by $u_{ij} := \chi_i(g_j)/\sqrt{c_j}$

where $c_j = |\mathrm{Cent}(g_j)| = |G|/|\mathrm{Cl}(g_j)|$

Then U is unitary

Proof: $\langle i\text{th row of } U, j\text{th row of } U\rangle$

$$= \sum_{k=1}^{s} u_{ik}\overline{u_{jk}} = \sum_{k=1}^{s} \frac{1}{c_k}\chi_i(g_k)\overline{\chi_j(g_k)}$$

$$= \sum_{k=1}^{s} \frac{|\mathrm{Cl}(g_k)|}{|G|}\chi_i(g_k)\overline{\chi_j(g_k)}$$

$$= \sum_{k=1}^{s}\sum_{g\in \mathrm{Cl}(g_k)} \frac{1}{|G|}\chi_i(g)\overline{\chi_j(g)}$$

$$= \frac{1}{|G|}\sum_{g\in G}\chi_i(g)\overline{\chi_j(g)}$$

$$= \langle \chi_i, \chi_j\rangle = \delta_{ij}$$

$\Rightarrow$ U is unitary

Theorem (Second Orthogonality Relations):

$\chi_1, \dots, \chi_s$ – irreducible characters

(1) $\forall g \in G$

$$\sum_{k=1}^{s} \chi_k(g)\overline{\chi_k(g)} = |\mathrm{Cent}(g)| = \frac{|G|}{|\mathrm{Cl}(g)|}$$

(2) $\forall g, h \in G$, g and h <u>not</u> conjugate

$$\sum_{k=1}^{s} \chi_k(g)\overline{\chi_k(h)} = 0$$

↳ columns of character table are pairwise orthogonal

Proof: $U = (u_{ij}) = \left(\chi_i(g_j)/\sqrt{c_j}\right)$

unitary $\Rightarrow$ columns are O.N.B of $\mathbb{C}^n$

Example : Character table of S_4

24	1	6	3	6	8	$\lvert cl(g) \rvert$
	e	$(1\,2)$	$(1\,2)(3\,4)$	$(1\,2\,3\,4)$	$(1\,2\,3)$	
χ_1	1	1	1	1	1	
χ_2	1	-1	1	-1	1	
χ_3	2	0	2	0	-1	
χ_4	3	1	-1	-1	0	
χ_5	3	-1	-1	1	0	
	24	4	8	4	3	$\lvert C_G(g) \rvert$